

Bekenstein Bound and Spectral Geometry

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In this letter it is proposed to study the Bekenstein's $\xi(4)$ calculation of the S/E bound for more general geometries. It is argued that, using some relations among eigenvalues obtained in the context of Spectral Geometry, it is possible to estimate $\xi(4)$ without an exact analytical knowledge of the spectrum. Finally it is claimed that isospectrality can define a class of domains with the same ratio S/E .

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I. INTRODUCTION.

Since the pioneer works of Bekenstein[1] and Hawking[2] about the gravitational entropy a substantial amount of work has been done trying to understand the amazing connection between the area, a geometrical quantity, and the entropy, a thermodynamic one. Within this context, Bekenstein also proposed[3] the existence of a universal bound of magnitude $2\pi R$ to the entropy-to-energy ratio S/E of an arbitrary system of effective radius R , or

$$S/E \leq 2\pi R. \quad (1)$$

Originally, the bound was deduced by considering a gedanken experiment of lowering the system into a black hole and demanding this process to satisfy the generalized second law of thermodynamics. On the other hand, one expects that there must be a limit to the entropy that can be placed in a system of finite size whose energy is limited. This is suggested by the limited phase space available to the components of such a system.

Besides that, Bekenstein himself proposed a explicit method to calculate the ratio S/E for fields inside symmetric cavities in two dimensions like the square, the rectangle and also in three dimension for fields inside the sphere, the cube, etc[4]. This calculation was done using the known spectrum (eigenvalues) for these geometries. Obviously the number of examples was limited to the few cases where it is known the analytic form of the spectrum.

In this paper we will show some interesting results coming from Spectral Geometry that allow to generalize the above mentioned Bekenstein method to more general geometries. Specifically, we will present some useful relations among the eigenvalues for generic domains that we will use to estimate the S/E ratio for a particular case. Finally we will illustrate how the connections between the Bekenstein's proposal and Spectral Geometry help to set more clearly the reasons behind the Bekenstein bound.

II. BEKENSTEIN APPROACH

In the work [5] was shown that if the cavity confining the system is circumscribed by a sphere of radius R then the microcanonical entropy $S(E) = \ln \Omega(E)$ obeys

$$S/E \leq [24\xi(4)]^{1/4}, \quad (2)$$

where $\xi(k)$ is the ξ -function

$$\xi(k) = \sum_i g_i \omega_i^{-k} \quad (3)$$

for the sphere, $\{\omega_i\}$ is the discrete one-particle energy with zero-modes excluded and g_i represents the degeneracy of the i -th level. Since for the sphere we have $\xi(4) \sim R^4$ the bound (1) follows from (2) provided $R^{-4}\xi(4)$ is appropriately bound from above. The later was verified in [5] for various types of free fields satisfying Dirichlet or Neumann conditions.

For the sake of simplicity we show the case of the scalar field inside and sphere. The solutions of the scalar equation which are harmonic in time may be found only for discrete eigenfrequencies ω_i which arise from the eigenvalue problem defined by

$$\nabla^2 \phi = -\omega^2 \phi, \quad (4)$$

together with the Dirichlet boundary conditions for ϕ . In this case the solutions are $j_{n,l}(\omega r)Y_{lm}(\theta, \phi)$, where j_l is the standard spherical Bessel function of order l . The boundary conditions then demands that ωR be a positive zero of j_l . Hence the spectrum is

$$\omega_{nl} = j_{n,l} R^{-1}, \quad n = 1, 2, \dots; \quad l = 0, 1, \dots, \quad (5)$$

where $j_{n,l}$ is the n th positive zero of $j_l(x)$, the degeneracy is $2l + 1$. The lowest eigenfrequency is $\omega_{10} = \pi/R$. With this at hand it is possible to calculate the analytical approximation to $\max(S/E)$ (for $R = 1$), that is

$$\xi(4)_{sphere}^{1/4} = 0.452. \quad (6)$$

Similar computations were done for different cavities in one, two and three dimensions[4].

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Another output of the works[4],[5] was the proof of a local theorem on the ξ function. A precise statement of this is that as a given cavity S is deformed into another one Σ entirely contained within it, all the eigenvalues increase and, therefore, the function ξ is smaller for Σ .

Related to the previous result is the existence of a lower bound for the lowest eigenvalue of the scalar field in an arbitrary odd-shaped cavity C that is circumscribed by a sphere of radius R , i.e.,

$$\omega_1 > \pi/R, \quad (7)$$

where the right hand side of the inequality is the first eigenvalue of the scalar field in the sphere [5]. Bekenstein obtained this result by applying the Rayleigh-Ritz principle to the eigenvalue equation (4).

III. SCALAR FIELDS IN GENERAL MANIFOLDS

In general, for any geometry, the one particle spectrum is poorly known, so in these situations it is not possible to calculate explicitly $\xi(4)$. Fortunately, in the last years the mathematicians working in the area of Spectral Geometry have shed light on this problem obtaining interesting results about the relation among the eigenvalues [6],[7]. Therefore in this section we will review some of the main inequalities for the eigenvalues of the Laplacian on bounded domains in Euclidean space. Our attention will be focused in the Dirichlet Laplacian or fixed membrane eigenvalue problem, i.e., the problem[6]

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (8)$$

where Ω is a bounded domain in Euclidean space \mathbb{R}^n and $\partial\Omega$ is its boundary (To avoid confusions with the previous notation it is worthy to point out that $\lambda_i = \omega_i^2$). It is well known that this problem has a real and purely discrete spectrum $\{\lambda_i\}_{i=1}^\infty$, satisfying,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots \lambda_n \dots \rightarrow \infty. \quad (9)$$

Here each eigenvalue is repeated according to its multiplicity.

In general, to solve the problem (8) is a difficult task and exact analytical solutions can be obtained just for some domains. However, there are some techniques that allows to obtain information about the bounds and relations satisfied by the eigenvalues. In the following lines we will present some of these results. Initially we show the Rayleigh-Ritz inequality, which gives a simple way to bound eigenvalues from above based on trial functions, i.e

$$\lambda_1(\Omega) = \inf_{\varphi \in D(-\Delta)} \frac{\int_{\Omega} \varphi(-\Delta\varphi)}{\int_{\Omega} \varphi^2}, \quad (10)$$

where φ is a real trial function in the domain of $-\Delta$. This kind of bound can be extended to higher eigenvalues by imposing orthogonality conditions on the class of trial functions used.

One of the earliest isoperimetric inequalities for an eigenvalue is certainly that for the first eigenvalue of the Dirichlet Laplacian and takes the form:

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) \quad \text{for } \Omega \subset \mathbb{R}^n, \quad (11)$$

with equality if and only if Ω is a ball, i.e., $\Omega = \Omega^*$. This is known as the Faber-Krahn Inequality.

The next isoperimetric result is for the quotient between the first two eigenvalues λ_1 and λ_2 . In 1955 and 1956 Payne, Polya and Weinberger(henceforth PPW) proved that[8]

$$\left. \frac{\lambda_2}{\lambda_1} \right|_{\Omega} \leq 3 \quad \text{for } \Omega \subset \mathbb{R}^2, \quad (12)$$

and conjectured that

$$\left. \frac{\lambda_2}{\lambda_1} \right|_{\Omega} \leq \left. \frac{\lambda_2}{\lambda_1} \right|_{\text{disk}} = \frac{j_{1,1}^2}{j_{0,1}^2} \approx 2.5387, \quad (13)$$

with equality if and only if Ω is a disk, $j_{p,k}$ denotes the k^{th} positive zero of the Bessel function. The analogous results for higher dimensions are

$$\left. \frac{\lambda_2}{\lambda_1} \right|_{\Omega} \leq 1 + \frac{4}{n} \quad \text{for } \Omega \subset \mathbb{R}^n, \quad (14)$$

and the PPW conjecture

$$\left. \frac{\lambda_2}{\lambda_1} \right|_{\Omega} \leq \left. \frac{\lambda_2}{\lambda_1} \right|_{n\text{-ball}} = \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2}, \quad (15)$$

with equality if and only if Ω is an n -ball. This PPW conjecture was proved in the work [10].

The search for relations between the eigenvalues was extended to higher eigenvalues in the form of universal inequalities and in 1955 Payne, Polya and Weinberger proved also that

$$\lambda_{m+1} - \lambda_m \leq \frac{2}{m} \sum_{i=1}^m \lambda_i, \quad m = 1, 2, \dots \quad (16)$$

for $\Omega \subset \mathbb{R}^2$. This result extends to $\Omega \subset \mathbb{R}^n$ as

$$\lambda_{m+1} - \lambda_m \leq \frac{4}{mn} \sum_{i=1}^m \lambda_i, \quad m = 1, 2, \dots \quad (17)$$

The inequality (17) is called a universal inequality because it applies to all domains $\Omega \subset \mathbb{R}^n$, "universally"[9].

A stronger inequality was derived by Hile and Protter[11] who proved that

$$\sum_{i=1}^m \frac{\lambda_i}{\lambda_{m+1} - \lambda_i} \geq \frac{mn}{4} \quad \text{for } m = 1, 2, \dots \quad (18)$$

Note that (18) implies (17), since we can replace the λ_i in the denominator of (18) by λ_m to obtain (17).

More recently, Yang [12] derived the inequality

$$\sum_{i=1}^m (\lambda_{m+1} - \lambda_i) \left(\lambda_{m+1} - \left(1 + \frac{4}{n}\right) \lambda_i \right) \leq 0 \quad (19)$$

for $m = 1, 2, \dots$

This inequality will be referred as Yang's first inequality to distinguish it from a simpler inequality implied by it (to be called Yang's second inequality). Inequality (19) is an implicit bound for λ_{m+1} , but an explicit bound can be derived from it by observing that its left hand side is just quadratic in λ_{m+1} . Therefore, taking the larger root and using the Cauchy-Schwarz inequality allow us to arrive at the Yang's second inequality

$$\lambda_{m+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{m} \sum_{i=1}^m \lambda_i. \quad (20)$$

This inequality is clearly stronger than the PPW inequality, since it results from replacing the λ_m by the average of the first m eigenvalues and λ_m is certainly larger than or equal than the average. Thus, we conclude that both of Yang's inequalities are stronger than the PPW inequality. On the other hand it can be proved also that the Yang inequality is stronger than the HP inequality[9]. This lead us to the following relations

$$\text{Yang 1} \implies \text{Yang 2} \implies \text{Hile - Protter} \implies \text{PPW} . \quad (21)$$

Although other interesting relations among the eigenvalues can be found in the literature those showed before are useful enough for our purposes.

A. Estimation of $\xi(4)$ for scalar fields in a deformed spherical cavity

In the first section was mentioned that the calculation of $\xi(4)$ for the scalar fields was done for various symmetric domains. Our purpose here is to give an estimate of this quantity for a domain obtained from a slight deformation of an sphere. In order to do that we will do some plausible considerations and use the relations among the eigenvalues presented in the previous section.

Our first consideration is that in this domain there is not degeneracy, therefore $g_i = 1$. This can be seen as a consequence of the deformation of the sphere that breaks all its symmetries. Now, taking into account the Faber-Krahn inequality(11) we assume, for example, that the first eigenvalue of the domain under study is 1 percent bigger than the first eigenvalue of the corresponding spherical problem, i.e, $j_{1/2,1} = \pi$. In order to get the second eigenvalue we could use the PPW inequality (15)

$$\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1} \Big|_{3\text{-ball}} = \frac{j_{3/2,1}^2}{j_{1/2,1}^2} = 2.04484. \quad (22)$$

To keep this inequality safe the second eigenvalue of the spherical problem can not be modified in an amount equal or superior to 1 percent. Therefore we assume a modification of 0.9 percent. Then the quotient between the two first eigenvalues is

$$\frac{\lambda_2}{\lambda_1} = 2.04080 < 2.04484. \quad (23)$$

For the higher eigenvalues we will use a modification of the Yang's second inequality(20). Actually we modify the relation among the eigenvalues substituting the factor $(1 + \frac{4}{3})$ by 2.04080. Therefore

$$\lambda_{m+1} = (2.04080) \frac{1}{m} \sum_{i=1}^m \lambda_i. \quad (24)$$

Using these assumptions and relations we are ready to calculate $\xi(4)$ for this domain, giving

$$\xi(4)_{dom}^{1/4} = \left(\sum_i \frac{1}{\lambda_i^2} \right)^{1/4} = 0.3536 . \quad (25)$$

This value is about 78 percent of the value obtained for the sphere. At this point would be interesting to note that if we calculate again $\xi(4)$ for the sphere and we neglect the degeneracies $2l + 1$ the result is

$$\xi(4)_{sphere}^{1/4} = 0.3586 \text{ for } g_i = 1 . \quad (26)$$

Therefore $\xi(4)_{dom}^{1/4}$ would be 98.59 percent of $\xi(4)_{sphere}^{1/4}$ (assuming $g_i = 1$). Consequently, we can conclude that in the case of a slight deformation of the cavity, from the spherical symmetry, the main cause in the decrease of $\xi(4)$ is due to the lost of degeneracies in the eigenvalues.

Obviously that, for a real case, we would need to know a clear relation between the degree of deformation of the geometry and the change in the eigenvalues. In this case the crucial point would be to do the best estimation of the first eigenvalue. Doing that we could obtain an acceptable estimation of $\xi(4)$.

IV. ON HEARING THE SHAPE OF A DRUM AND ISOSPECTRALITY

At this point it is not difficult to imagine that the knowledge of the spectrum of a determined domain can help us to gain essential information of the system. Already in 1911 Herman Weyl proved that the area of a plane domain is determined by its spectrum[13]. Some years later the Swedish mathematician Ake Pleijel also proved that it is possible to obtain the length of the boundary of the domain [14] from the spectrum as well. These relations between the spectrum and the geometrical properties of the domain can be shown explicitly using the trace of the heat kernel, ie,

$$Z(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}, \quad (27)$$

where $\{\lambda_n\}$ are the eigenvalues of the Laplace operator. If the domain M has a smooth boundary, $Z(t)$ has an asymptotic expansion for a small positive t , given by the Minakshisundaram-Pleijel formula,

$$Z(t) = \frac{1}{4\pi t} \sum_{k=0}^{\infty} D_k t^{k/2}, \quad (28)$$

where coefficients D_k reflect the geometric nature of the domain M . Particularly,

$$D_0 = \text{Area}(M), \quad (29)$$

$$D_1 = -\frac{\sqrt{\pi}}{2} \text{Length}(\partial M). \quad (30)$$

These interesting results led to the speculation that perhaps the shape of a plane domain (or more generally, of a Riemannian manifold) is audible. It is worthy to remember that if M is a domain in the Euclidean plane then the Dirichlet eigenvalues of Δ are essentially the frequencies produced by a drumhead shaped like M . In this line, in a landmark paper [15], Mark Kac posed the question "Can one hear the shape of a drum?". In the case of a Riemannian manifold, the Kac's question can be paraphrased as: "Can one deduce the metric of the surface from the spectrum?". Until the moment the answer to this question is not known in sufficient detail. An affirmative answer is known to hold for several classes of surfaces and domains [16], [17]. However, this is not always true. One of the first examples to the negative is due to Milnor who proposed in 1964 two flat tori in \mathbb{R}^{16} , which he proved to be isospectral but not isometric. Since then, many other pairs of isospectral (counting multiplicities) yet not isometric systems were found. A general method for constructing isospectral, non-isometric manifolds has been

designed by Sunada [19]. Despite these advances in several dimensions the problem for plane regions remained open until 1991, when Carolyn Gordon, David Webb, and Scott Wolpert found examples of distinct plane "drums" which "sound" the same. Lately this was confirmed experimentally by the work of Sridhar and Kudrolli [21]. In the experiments they employed thin microwave cavities shaped in the form of two different domains known to be isospectral. Specifically, they verified the equality of at least 54 of the measured low-lying eigenvalues to a few parts in 10^4 . On the other hand Driscoll [22] showed a method to evaluate numerically the eigenvalues of polygonal regions.

On the light of the results presented above we could conclude that isospectral domains, in the case of scalar fields, have the same relation S/E . In other words, isospectrality allows to define a class of systems with the same ratio S/E . That is obviously clear from the form of $\xi(4)$. On the other hand geometric constraints are forced on isospectral manifolds and this fact could suggest that these domains have the same effective radius. Therefore we can conclude that these results coming from the field of Spectral Geometry support strongly the Bekenstein proposal. This conclusion could be reinforced by some results, recently found, that relate Information Theory and Spectral Geometry [23].

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